Solutions of homework # 3

A) The eigenvalues λ_i of an operator $\hat{A}$ can be computed solving the secular equation $\mathrm{Det}|\hat{A}-\lambda\hat{I}| = 0$, where $\hat{I}$ is the identity operator. Once we have the eigenvalues, we can compute the eigenvectors $|\psi_i\rangle$ using their definition $\hat{A}|\psi_i\rangle = \lambda_i|\psi_i\rangle$. It is easy to verify that the eigenvalues and eigenvectors of the operator $\hat{A}$ are:

$$\lambda_1 = E\,, \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{1}$$

$$\lambda_2 = -E\,, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{2}$$

Note, we have used the normalization condition for $|\psi_1\rangle$ and $|\psi_2\rangle$: $\langle\psi_i|\psi_i\rangle = 1$.

The operator $\hat{A}$ is Hermitian, because $A_{ij} = A^*_{ji}$. This means that the operator $\hat{A}$ can be associated with an observable quantity, i.e. its eigenvalues can be measured in the experiment. As you know from lectures, according to the postulates of quantum mechanics in any single measurement of a physical quantity that corresponds to a Hermitian operator (in our case we have $\hat{A}$ which is Hermitian), the only values that will be measured are the eigenvalues of that operator.

From the postulates of quantum mechanics you know that immediately after a measure of one of the eigenvalues of $\hat{A}$ the system collapses in the eigenfunction associated to the measured eigenvalue. In other words, if λ_1 is measured then the system collapses to $|\psi_1\rangle$, if λ_2 is measured then the system collapses to $|\psi_2\rangle$.

B) Let us verify whether the operators $\hat{A}$ and $\hat{H}$ commute.

$$[\hat{A},\hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A}\,, \tag{3}$$

$$EE_0\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - EE_0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = EE_0\begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix}. \tag{4}$$

This means that $[\hat{A},\hat{H}] \neq 0$. Therefore, the operators $\hat{A}$ and $\hat{H}$ **do not** share a common basis of eigenfunctions. Indeed, the eigenvalues and eigenfunctions of $\hat{H}$ are

$$\tilde{\lambda}_1 = E_0\,, \qquad |\tilde{\psi}_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5}$$

$$\tilde{\lambda}_2 = -E_0\,, \qquad |\tilde{\psi}_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6}$$

and, thus, the eigenstates of $\hat{A}$ are not the eigenstates of $\hat{H}$, and vice versa.

Exercise 2

Since A is a Hermitian operator, its eigenstates $\{\psi_{i=1,3}\}$ form a complete basis set for the Hilbert space. Therefore any state $|\Phi\rangle$ can be written as a linear combination of the basis vectors:

$$|\Phi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + c_3|\psi_3\rangle \tag{7}$$

From the postulates of quantum mechanics we know that a measure of A on the generic state $|\Phi\rangle$ can only give one of the eigenvalues ϵ_i of the operator and that the probabilities are simply given by

$$p_i = |\langle\psi_i|\Phi\rangle|^2 = |c_i|^2, \tag{8}$$

where in the last identity we used the definition of $|\Phi\rangle$ in Eq. (7) and the fact that the basis vectors $|\psi_i\rangle$ are orthogonal and normalized ($\langle\psi_i|\psi_j\rangle = \delta_{ij}$).

a) We can now look for a state such that $p_1 = 1/2$ (50% of probability to obtain ϵ_1 from a measure of A) and $p_2 = p_3 = 1/4$ (25% for both ϵ_2 and ϵ_3). These conditions determine the values of the square modulus of the coefficients c_i:

$$|c_1|^2 = \frac{1}{2}, \qquad |c_2|^2 = \frac{1}{4}, \qquad |c_3|^2 = \frac{1}{4} \tag{9}$$

This equation tells us that imposing a given set of probabilities only fixes the square modulus of the coefficient describing our state. If we think to our complex coefficient as a vector in the complex plane, the condition on the square modulus fixes the "length" of the vector but not its "direction". Indeed recalling that the coefficients are in general complex number, the most general solution of Eq. (9) is

$$\begin{aligned} |c_1|^2 = \frac{1}{2} \quad &\Rightarrow \quad c_1 = \frac{1}{\sqrt{2}}e^{i\vartheta_1} \\ |c_2|^2 = \frac{1}{4} \quad &\Rightarrow \quad c_2 = \frac{1}{2}e^{i\vartheta_2} \\ |c_3|^2 = \frac{1}{4} \quad &\Rightarrow \quad c_3 = \frac{1}{2}e^{i\vartheta_3} \end{aligned} \tag{10}$$

with ϑ_i arbitrary real numbers. The most generic state can therefore be written as

$$|\Phi''\rangle = \frac{1}{\sqrt{2}}e^{i\vartheta_1}|\psi_1\rangle + \frac{1}{2}e^{i\vartheta_2}|\psi_2\rangle + \frac{1}{2}e^{i\vartheta_3}|\psi_3\rangle \tag{11}$$

If we set $\vartheta_i = 2\pi$ for $i = 1, 2, 3$, we obtain one possible solution, i.e. $c_1 = 1/\sqrt{2}$, $c_2 = c_3 = 1/2$, and the explicit expression of the state becomes

$$|\Phi\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{2}\left(|\psi_2\rangle + |\psi_3\rangle\right) \tag{12}$$

b) Let us now consider the state $|\Phi'\rangle = e^{i\vartheta}|\Phi\rangle$ with ϑ some real number. Let us call p'_i the probabilities to obtain ϵ_i from a measure of A on this new state

$$\begin{aligned} p'_1 &= |\langle\psi_1|\Phi'\rangle|^2 = |\langle\psi_1|\Phi\rangle e^{i\vartheta}|^2 = |\langle\psi_1|\Phi\rangle|^2|e^{i\vartheta}|^2 = |\langle\psi_1|\Phi\rangle|^2 = p_1 \\ p'_2 &= |\langle\psi_2|\Phi'\rangle|^2 = |\langle\psi_2|\Phi\rangle e^{i\vartheta}|^2 = |\langle\psi_2|\Phi\rangle|^2|e^{i\vartheta}|^2 = |\langle\psi_2|\Phi\rangle|^2 = p_2 \\ p'_3 &= |\langle\psi_3|\Phi'\rangle|^2 = |\langle\psi_3|\Phi\rangle e^{i\vartheta}|^2 = |\langle\psi_3|\Phi\rangle|^2|e^{i\vartheta}|^2 = |\langle\psi_3|\Phi\rangle|^2 = p_3 \end{aligned} \tag{13}$$

c) The expectation value of a generic operator B on the state $|\Phi'\rangle$ can be easily computed

$$\langle B \rangle = \langle \Phi' | B | \Phi' \rangle = \langle \Phi | e^{-i\vartheta} B e^{i\vartheta} | \Phi \rangle = \langle \Phi | B e^{-i\vartheta} e^{i\vartheta} | \Phi \rangle = \langle \Phi | B | \Phi \rangle \tag{14}$$

where we used the fact that a complex number always commutes with an operator. In our particular case $e^{-i\vartheta} B = B e^{-i\vartheta}$. The result above implies that the measure of any observable of the system will give the same result whether our system is in the state $|\Phi\rangle$ or $|\Phi'\rangle$. Therefore, according to the particular choice of the real number ϑ, we have an infinite number of equivalent states carrying the same physical information.